Limits Lecture 1: The idea

Since prehistoric days, when we were still living in caves, the hardest number to digest was 0.

Think about it! You come home to your wife.

She asks you: How many buffalo did you catch!?

5? Wonderful!

And what are you going to tell her if you return empty handed?

Don't come home having brought nothing!

Tell her proudly that you brought 0 buffalo instead, for 0 isn't nothing.

0 is the additive identity of the real numbers

0 is defined by $0+x = x+0 = x$ for all numbers x

$0 \cdot x = 0$ for all x

$$0 \cdot x = (0+0)x = 0 \cdot x + 0 \cdot x$$

Thus

$$0 = 0 \cdot x - 0 \cdot x = 0 \cdot x + 0 \cdot x - 0 \cdot x$$

or

$$0 = 0 \cdot x$$

<u>Division by 0 is undefined</u>

For a number like 5, $\frac{1}{5}$ is its multiplicative inverse. That is

$$5 \cdot \frac{1}{5} = 1.$$

$\frac{1}{0}$ would have to be the number that, when multiplied by 0, generates 1.

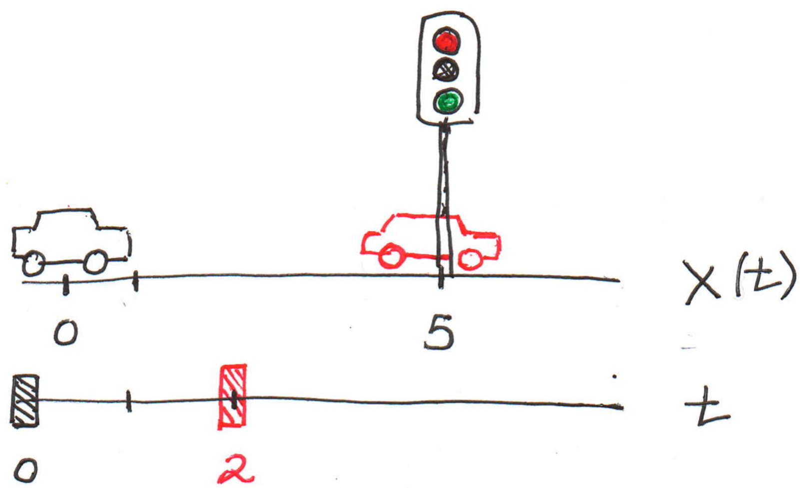
But $\quad 0 \cdot \frac{1}{0} = 0 \cdot (\text{number}) = 0$

Thus, there cannot be a number $\frac{1}{0}$ that multiplies 0 to 1.

But what do we do if we desperately need to divide by 0?

The Velocity Problem

You are presented with a video of a moving car and asked to determine the speed of this car as it passes the traffic light.



By pressing pause and play, you obtain the following table

t	$X(t)$
0	1
1	2
2	5
3	10

Can you come up with a formula for $X(t)$?

These numbers are consistent with the formula

$$X(t) = t^2 + 1$$

Now, lets try to better understand what is it that we have to find.

Q. what is velocity?

A. Velocity is change of position divided by change of time. More preceisly, an object undergoing uniform motion has velocity which can be computed as a ratio of any distance traveled, divided by the time required to travel it. For example, a runner moving 6 mi in one hr is expected to cover 3 mi in ½ hr and 12 mi in 2 hrs because

$$\frac{6}{1} = \frac{3}{\left(\frac{1}{2}\right)} = \frac{12}{2} = 6 \ {}^{mi}/_{hr}.$$

We can try to calculate this for the moving car

We want to know how fast the car was driving past the traffic light.

We can try $\frac{X(2) - X(0)}{2-0} = \frac{5-1}{2}$

$= 2$.

However $\frac{X(2) - X(1)}{2-1} = \frac{5-2}{1} = 3$

Velocities don't match. Car isn't traveling in uniform constant motion.

Q. Does the car have a velocity? In particular Does the car have velocity at $t=2$? When the video is paused, does the car have velocity? How do you know?

A. We are really trying to figure out the instanteneous velocity at $t=2$. This means, we're trying to find out what the speedometer of the car is displaying at $t=2$.

Although the car isn't traveling at constant speed, our experience tells us that velocity requires time to accelerate or decelerate.

<u>Idea:</u> If we don't give any time to ellapse, we'll get the velocity at $t=2$.

$$\frac{x(2)-x(2)}{2-2} = \frac{0}{0} \quad !!!$$

<u>Idea:</u> Plugging in 2 twice doesn't work, because we end up dividing by 0. Lets try to plug in a "fake" 2, $\hat{2}$:

$$\frac{x(\hat{2})-x(2)}{\hat{2}-2} = \frac{[(\hat{2})^2+1]-[2^2+1]}{\hat{2}-2}$$

$$= \frac{(\hat{2})^2-2^2}{\underbrace{\hat{2}-2}_{\text{fake } 0}} = \frac{\cancel{(\hat{2}-2)}(\hat{2}+2)}{\cancel{(\hat{2}-2)}} = \hat{2}+2$$

≈ 4.

Notice that we could divide out $\hat{2}-2$, because although division by a real zero isn't defined, division by a counterfeit zero is perfectly fine!

We predict that the instanteneous velocity at $t=2$ (the speedometer) displays $4 \frac{\text{unit distance}}{\text{unit time}}$.

In general, the speedometer indicates velocity $V(t)$:

$$V(t) = \frac{x(\hat{t})-x(t)}{\hat{t}-t} = \frac{[(\hat{t})^2+1]-[t^2+1]}{\hat{t}-t} =$$

$$= \frac{\hat{t}^2-t^2}{\hat{t}-t} = \frac{\cancel{(\hat{t}-t)}(\hat{t}+t)}{\cancel{(\hat{t}-t)}}$$

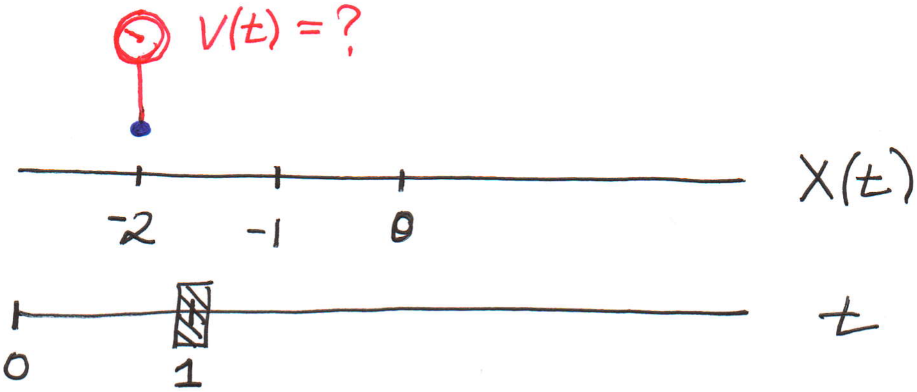
$$= \hat{t}+t \approx 2t.$$

Thus, we predict $V(t) = 2t$.

Ex. A particle moves so that its position at time t is given by $x(t) = -4t^2+2$

(a) Find its velocity as a function of t.

(b) What is this velocity at $t=1$? In which direction is the particle moving?



<u>Solution:</u>

(a) The idea is to select $\hat{t}$ very close to t (a conterfeit t) and compute

$$V(t) = \frac{X(\hat{t}) - X(t)}{\hat{t} - t} = \frac{\left[-4(\hat{t})^2 + 2\right] - \left[-4t^2 + 2\right]}{\hat{t} - t}$$

$$= \frac{-4(\hat{t}^2 - t^2)}{(\hat{t} - t)} = \frac{-4\cancel{(\hat{t} - t)}(\hat{t} + t)}{\cancel{(\hat{t} - t)}}$$

$$= -4(\hat{t}+t) \approx -4(2t) = -8t.$$

Thus $V(t) = -8t$.

<u>Remark:</u> If $\hat{t}$ is nearly t, then speedometer didn't have the time to change in between positions $X(t)$ and $X(\hat{t})$. The particle was traveling at constant speed. (Or essentially constant speed)

(b) The instanterious velocity at $t = 1$ is

$V(1) = -8 \cdot 1 = -8$

If X is in meters and t in seconds, then the particle is travelling -8 m/sec to the left.

We will return to this topic when we introduce the derivative. For now, let's try to understand this conterfeit number a bit better.

In calculus we have two notions of evaluating a function at the number c:

$f(c)$ and $\lim\limits_{x \to c} f(x)$.

In the first case one simply plugs in the value c, whereas $\lim\limits_{x \to c} f(x)$ means "plug in many values closer and closer to c, but <u>never</u> c itself."

<u>Ex.</u> Let $f(x) = 2x$

Then $f(-1) = -2$

$\lim\limits_{x \to -1} f(x)$ is computed as follows:

x	$f(x)$
-1.1	$2(-1.1) = -2.2$
-1.01	$2(-1.01) = -2.02$
-1.001	$2(-1.001) = -2.002$
$\downarrow$	$\downarrow$
-1	-2

Thus $\lim\limits_{x \to -1} f(x) = -2 = f(-1)$

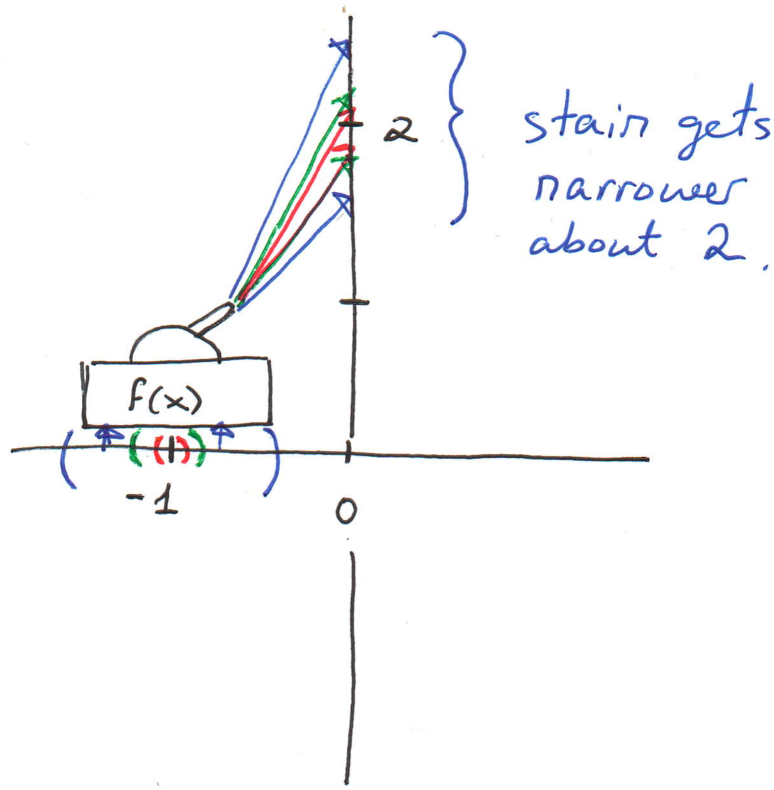
<u>Ex.</u> Let $f(x) = \begin{cases} x^2+1 & \text{if } x \neq -1 \\ 3 & \text{if } x = -1 \end{cases}$

Then $f(-1) = 3$. To compute $\lim_{x \to -1} f(x)$ construct

x	$f(x)$
-0.99	$(-0.99)^2+1 = 1.9801$
-0.999	$(-0.999)^2+1 = 1.998001$
-0.9999	$(-0.9999)^2+1 = 1.99980001$
$\downarrow$	$\downarrow$
-1	2

Notice that in this case $\lim_{x \to -1} f(x) = 2 \neq f(-1)$.

In the table we only selected some sequence of values approaching -1. Rather than imagining the calculation of limits as done over a table of values, think of the following watergun analogy: Think of the x-axis as a body of water, the function $f(x)$ as a water cannon, and the y-axis as a wall on which the water particles land.

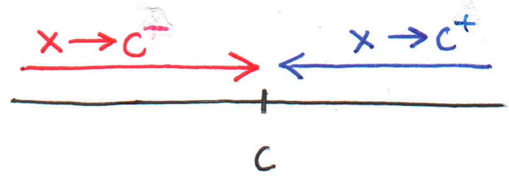


(a) Evaluating $f(-1)$ is the same as plugging one water particle $x=-1$.

(b) Evaluating $\lim_{x \to -1} f(x)$ is the same as plugging in all values (except -1) in the viscinity of -1 and then narrowing the pipe.

This water cannon analogy is useful in defining limits rigorously.

One-sided limits



$\lim_{x \to c^+} f(x)$ means that we plug in only values that are bigger than c

$\lim_{x \to c^-} f(x)$ means that we plug in only values below c.

Ex. let $f(x) = \begin{cases} 2x & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ 3x & \text{if } x > 3 \end{cases}$

Compute

(a) $f(3)$

(b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$

Solution:

(a) $f(3) = 5$

(b)

x	$f(x)$
2.9	$2(2.9)$
2.99	$2(2.99)$
2.999	$2(2.999)$
$\downarrow$	$\downarrow$
3	6

We can also reason as follows:

If $x < 3$ we use the formula $f(x) = 2x$

so if we plug a number similar to 3, you obtain $2 \cdot 3$.

thus $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2x = 2(3^-) = 6^- = 6.$

(c)

x	$f(x)$
3.1	$3(3.1)$
3.01	$3(3.01)$
3.001	$3(3.001)$
$\downarrow$	$\downarrow$
3	9

thus $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 3x = 9^+ = 9$

Def: We say $\lim_{x \to c} f(x)$ exist if

1) $\lim_{x \to c^-} f(x)$ exists

2) $\lim_{x \to c^+} f(x)$ exists

3) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

So, for example, in the previous examble

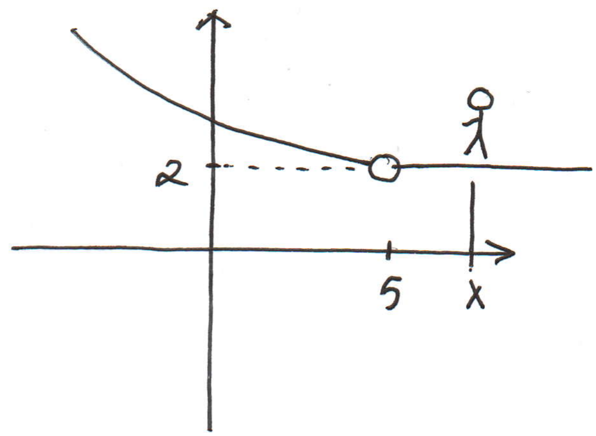
$\lim_{x \to 3} f(x) = \emptyset$ (does not exist)

because $\lim_{x \to 3^-} f(x) = 6 \neq 9 = \lim_{x \to 3^+} f(x)$.

Limits from Graphs

We are familiar with graphs depicting the behavior of a function. Imagine that the graph of $y=f(x)$ is a 1-D landscape on which we can walk a doll.

Ex.



Find (a) $f(5)$

(b) $\lim_{x \to 5^-} f(x)$, $\lim_{x \to 5^+} f(x)$

(c) $\lim_{x \to 5} f(x)$.

Solution:

(a) There is no solid dot at $x=5$ so the doll cannot stand there. $f(5) = \emptyset$

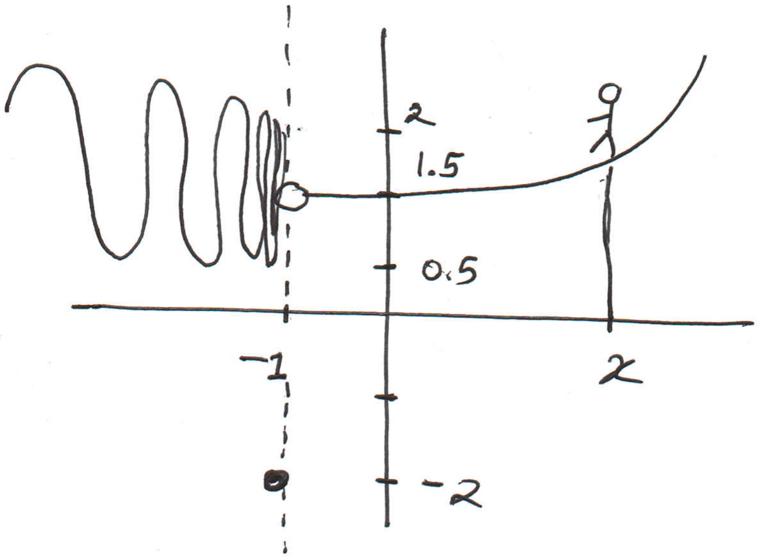
(b) $\lim_{x \to 5^-} f(x) = 2^+ = 2$

$\lim_{x \to 5^+} f(x) = 2$

(c) Since the one-sided limits of (b) agree

$\lim_{x \to 5} f(x) = 2$.

Ex.



Find

(a) $f(-1)$

(b) $\lim\limits_{x \to -1^-} f(x)$, $\lim\limits_{x \to -1^+} f(x)$

(c) $\lim\limits_{x \to -1} f(x)$

Solution:

(a) $f(-1) = -2$

(b) $\lim\limits_{x \to -1^+} f(x) = 1.5^+ = 1.5$

(c) $\lim\limits_{x \to -1^-} f(x)$ oscilates between 0.5 and 2

so $\lim\limits_{x \to -1^-} f(x) = \emptyset$.

Thus $\lim\limits_{x \to -1} f(x) = \emptyset$ as well.

Infinite limits

Ex. Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists

Solution:

x	$\frac{1}{x^2}$
$\pm\frac{1}{10}$	$\frac{1}{\left(\pm\frac{1}{10}\right)^2} = 100$
$\pm\frac{1}{100}$	$\frac{1}{\left(\pm\frac{1}{100}\right)^2} = 10000$
$\pm\frac{1}{1000}$	$\frac{1}{\left(\pm\frac{1}{1000}\right)^2} = 1000000$
$\downarrow$	$\downarrow$
0	∞

Def: We say that $\lim_{x \to c} f(x) = \infty$

(limit diverges to ∞ infinity) if, when we plug numbers closer and closer to c, the outputs grow without bound.

Remark: ∞ isn't a real infinity. It is also not a number. Rather ∞ expresses "growth without bound."

Since ∞ is not a number, $\lim_{x \to c} f(x) = \infty$ means that this limit doesn't exist.

Ex. Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$

Solution:

First $\frac{2x}{x-3} = 2\left(\frac{x-3+3}{x-3}\right) =$

$= 2\left(1 + \frac{3}{x-3}\right) = 2 + \frac{6}{x-3}$

$\lim_{x \to 3^+} \left(2 + \frac{6}{x-3}\right) = 2 + \frac{6}{0^+}$

$= 2 + \infty = \infty$

$\lim_{x \to 3^-} \left(2 + \frac{6}{x-3}\right) = 2 + \frac{6}{0^-} = 2 - \infty = -\infty$

Both limits do not exist.